이학박사 학위논문

Analysis on Simplicial Complex via Kirchhoff's laws

(키르히호프 법칙을 통한 단체 복합체 분석)

2019년 2월

서울대학교 대학원

수리과학부

오중석

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이 논문을 이학박사 학위논문으로 제출함

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Analysis on Simplicial Complex via Kirchhoff's laws

A dissertation
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of the requirements for the degree of
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by

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Abstract

Analysis on Simplicial Complex via Kirchhoff's laws

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In this thesis, we introduce the notion of Kirchhoff's conditions for a simplicial network (X, R) where X is a simplicial complex and R is a set of resistances for the top simplices, and prove the relation between Kirchhoff's conditions and i-dimensional weighted spanning tree number. Then, we can generalize effective resistance for simplicial network (X, R) where resistances are real valued.

Key words: Kirchhoff's conditions, Weighted spanning tree number, Resistor network, weighted cycle intersection matrix, Effective resistance
Student Number: 2012-20251

Contents

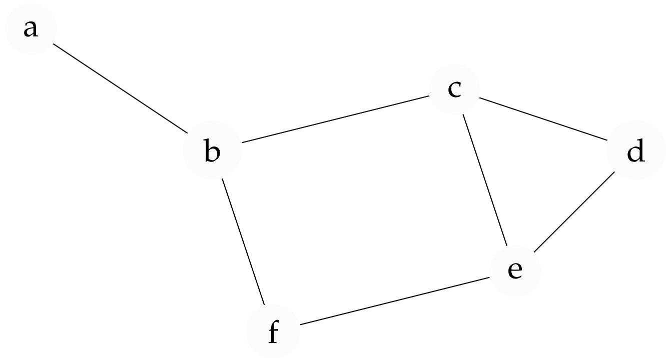
Chapter 1

Kirchhoff's Conditions on Graphs

1.1 Graphs and Spanning Trees

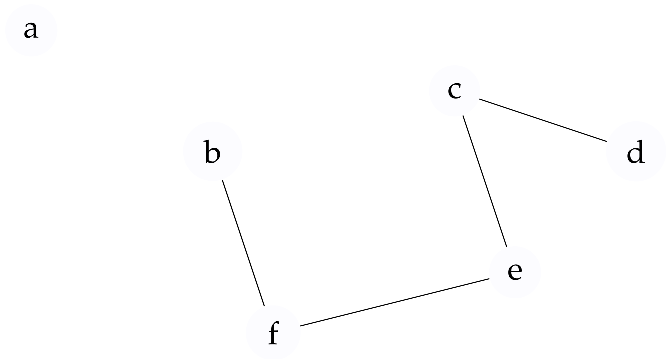
Definition (Graph)**.** A graph is an ordered pair G=(V, E), where V is a set and E is a set of 2-element subsets of V. We call elements of V as verticies and elements of E as edges.

Example 1.1. $V = \{a, b, c, d, e, f\}$ and $E = \{\{a,b\}, \{b,c\}, \{b,f\}, \{c,d\}, \{c,e\}, \{d,e\}, \{f,e\}\}$. It can be seen as below.



Definition (Subgraph)**.** Let G=(V, E) be a graph. For a subset S of V and a subset L of E, F=(S, L) is called a subgraph of G.

Example 1.2. Let G be the graph of Example 1.2. In case of $S = \{a,\ b,\ c,\ d,\ e,\ f\}$ and $L = \{\ \{b, f\},\ \{f, e\},\ \{e, c\},\ \{c, d\}\ \}$, It can be seen as below.



Definition (degree of vertex)**.** Given a graph G=(V, E). For v ∈ V, The degree of v is defined by $d(v)$=$|\{e \in E | v \in e\}|$.

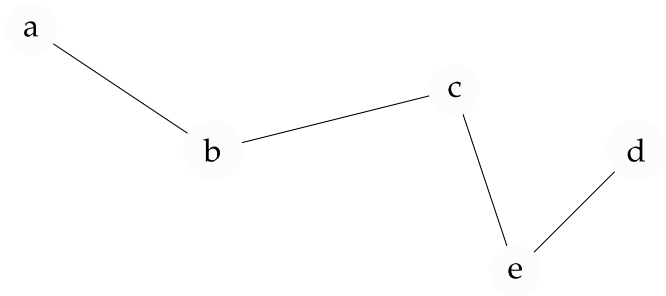
Example 1.3. Let G be the graph of Example 1.2. Then, $d(a) = 1$, $d(b) = 3$, $d(c) = 3$, $d(d) = 2$, $d(e) = 3$, $d(f) = 2$.

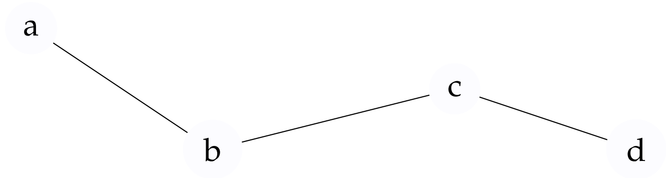
Definition (Neighborhood)**.** Given a graph G=(V, E). For a subset S of V, N(S)={ a ∈ V | { a, s } ∈ E, s ∈ S } is called the neighborhood of S in V.

Example 1.4. Let G be the graph of Example 1.2. Then, $N(a) = \{b\}$, $N(\{a, b\}) = \{b, f, c\}$.

Definition (Neighborhood sequence and Path)**.** Let a G=(V, E) be a graph. Start from an element a_1 of V, select an element a_i in $N(a_{i-1})$ recurrently. We call such a sequence $\{a_i\}$(i ∈ [n]) as neighborhood sequence of V. For any elements a and b in V, if there exists a neighborhood sequence of $\{a_i\}$(i ∈ [n]) for some n such that a_1=a and a_n=b, we say that $\{\{a_i, a_{i+1}\}\}$ is a path between a and b.

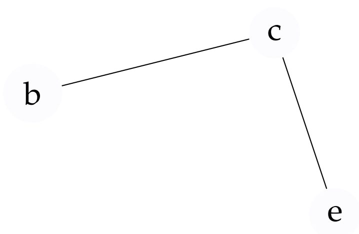
Example 1.5. Let G be the graph of Example 1.2. Then, $a_1 = a$, $a_2 = b$, $a_3 = c$, $a_4 = e$, $a_5 = d$ and $b_1 = a$, $b_2 = b$, $b_3 = c$, $b_4 = d$ are neighborhood sequences. Also, We can call $\{\{a_1, a_2\}, \{a_2, a_3\}, \{a_3, a_4\}, \{a_4, a_5\}\}$ and $\{\{b_1, b_2\}, \{b_2, b_3\}, \{b_3, b_4\}\}$ as paths between a and d.

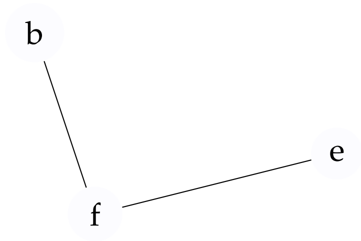


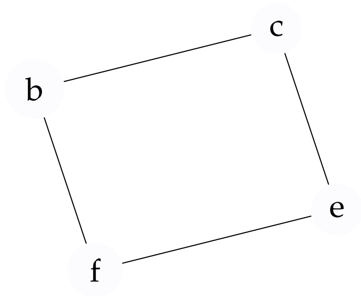


Definition (Cycle)**.** Given a graph G=(V, E). For any elements a and b in V, if there exist $\{a_i\}$, $\{b_i\}$, paths between a and b, such that $\{a_i\}\cap\{b_i\}$={a, b}, We call $\{\{a_i,\ a_{i+1}\}\} \cup \{\{b_i,\ b_{i+1}\}\}$ as cycle.

Example 1.6. Let G be the graph of Example 1.2. Consider two paths between b and e. One of them is $\{\{b,c\},\ \{c,e\}\}$. Another is $\{\{b,f\},\ \{f,e\}\}$. Then, the union of these two paths $\{\{b,c\},\ \{c,e\},\ \{b,f\},\ \{f,e\}\}$ is cycle.





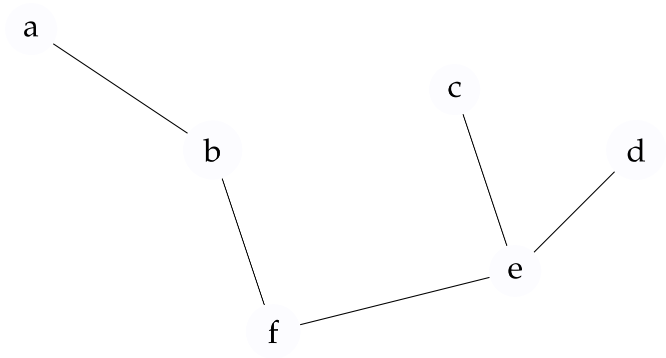


Definition (Spanning tree)**.** Let G=(V, E) be a graph. For a subgraph T=(V, L) of G, We call F as spanning tree if T satisfy following two conditions.

1.(Connected)For every pair (a, b) of V, There exists a path between a and b.

2.(Acyclic)there is no cycle in T.

Example 1.7. Let G be the graph of Example 1.2. Followings are spanning trees of G. At all cases, there exists path of every pair of V and there is no cycle.



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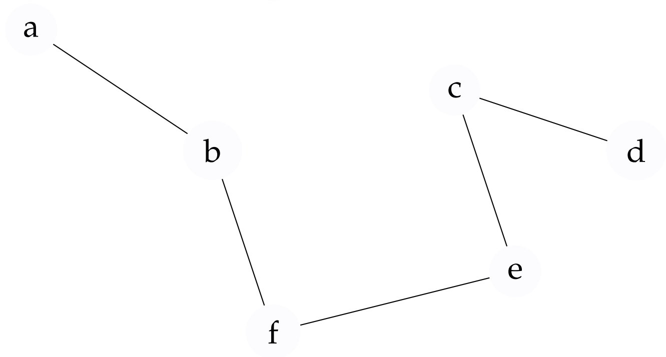

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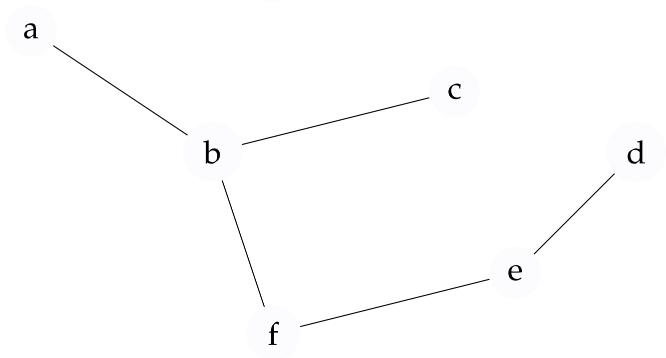

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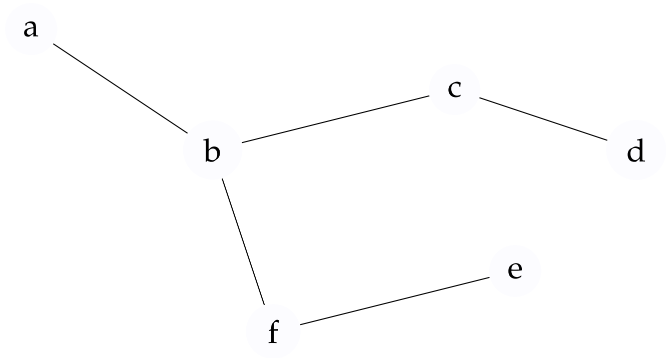
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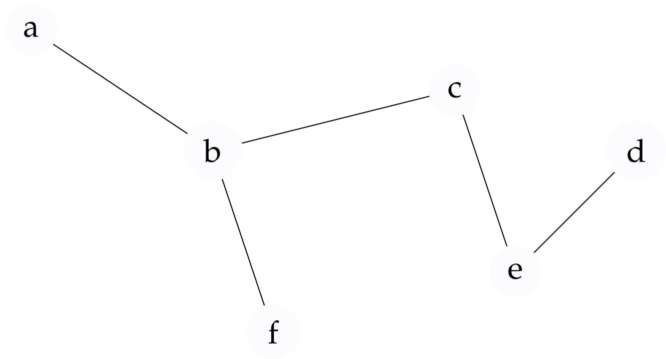
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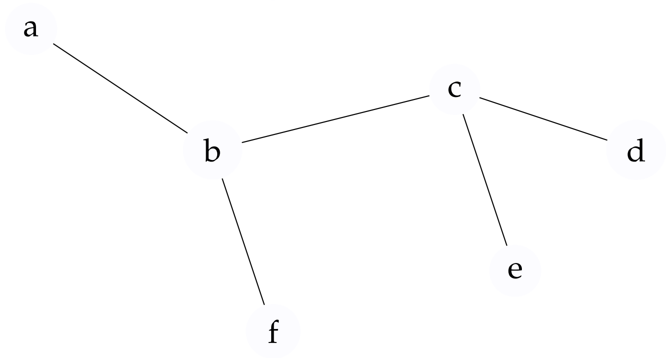
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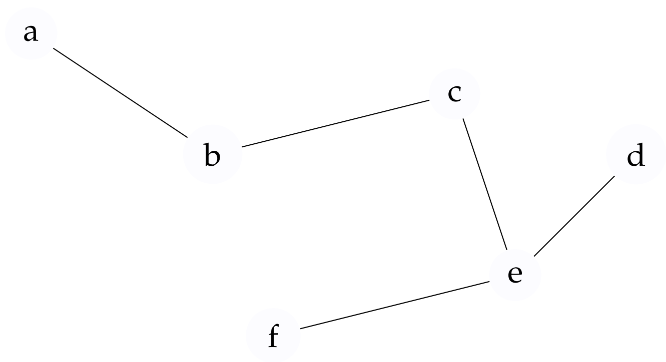
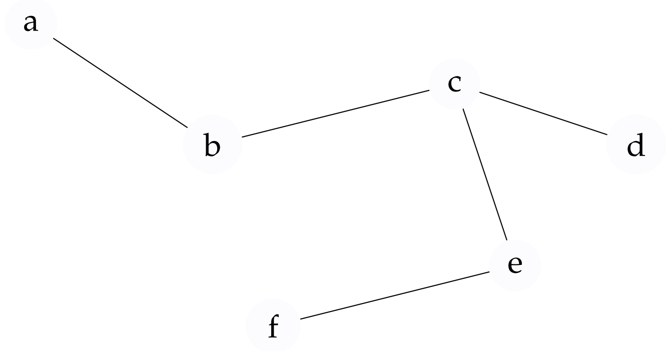
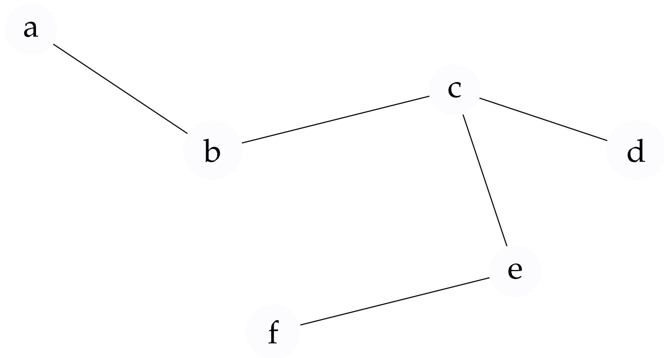


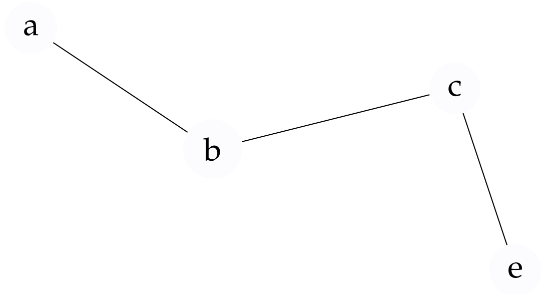


Proposition 1.8. *Given a graph G=(V, E) and a spanning tree T=(V, L) of G. For every pair of (a, b) of V, there exists a unique path between a and b of T.*

Proof. Suppose there exist two diffrent paths $\{\{a_i, a_{i+1}\}\}$, $\{\{b_k, b_{k+1}\}\}$ between a and b. Let s be the first number such that $a_{s+1} \neq b_{s+1}$(Since they are diffrent paths, there exists s). Also, Let e and e' be the first number such that $a_e = b_{e'}(e > s, e' > s$, Since they are paths from a to b, there exists e). Then, $\{a_i\}$(i ∈ [s, e]) and $\{b_i\}$(i ∈ [s, e']) are paths from a_s to $b_{e'}$ consisting a cycle. It's contradiction to T has no cycle. So, There exists a unique path for every pair of (a, b) of V. □

Example 1.9. Let G be the graph of Example 1.2. In case of the last spanning tree of Example 1.14, $\{\{a,b\},\ \{b,c\},\ \{c,e\}\}$ is the unique path between a and e. We can easily verify other paths.





Proposition 1.10. *Given a graph G=(V, E) and a spanning tree T=(V, L) of G. Then,* $|L| = n-1$ *where* $n = |V|$.

Proof. Consider $\{e_i\}$, labelling of L, such that $e_i \cap e_k \neq$ for some $k < i$(It's possible since T satisfy connected condition). Note that each e_l gives a new vertex a sequenticially to $\{e_i\}$ for i $\in [l-1]$. (If not, it means that T has a cycle. It's contradiction to T has acyclic condition). So, $|L| = n-1$. □

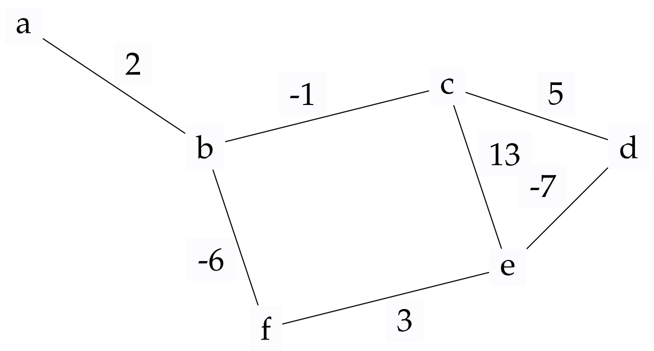
Example 1.11. Let G be the graph of Example 1.2. From Example 1.14, all spanning trees of G, We can verify that the number of edge is $5 = |V| - 1$.

Proposition 1.12. *Given a graph G=(V, E) and a spanning tree T=(V, L) of G. There exists a vertex v such that d(v)=1.*

Proof. Note that Since every edge e of E is 2-elements subset of V, e gives one degree to each of two vertices. So, by Proposition 1.9., the sum of d(v) of V are $2(n-1) = 2n-2$. If there is no v such that d(v)=1, it means that the sum of d(v) of V are larger than $2n-1$. It's contradiction to above statement. So, there exists v such that d(v)=1. □

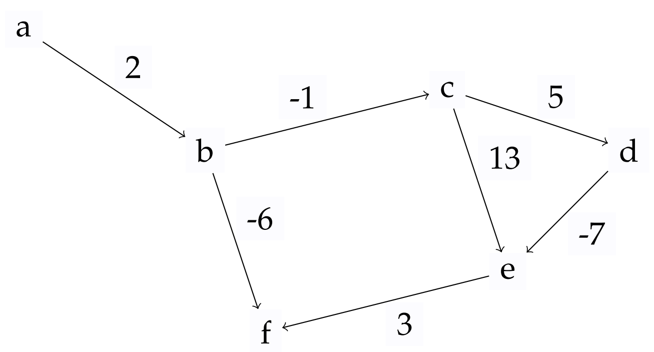
Definition (Weighted graph)**.** A weighted graph is a graph in which a complex number w_e is assigned to each edge e.

Example 1.13. $V = \{a,\ b,\ c,\ d,\ e,\ f\}$ and
$E = \{\ \{a,b\},\ \{b,c\},\ \{b,f\},\ \{c,d\},\ \{c,e\},\ \{d,e\},\ \{f,e\}\}$. If we give weight to each edges by 2 to $\{a,b\}$, -1 to $\{b,c\}$, 5 to $\{c,d\}$, -7 to $\{d,e\}$, 13 to $\{c,e\}$, 3 to $\{e,f\}$ and -6 to $\{b,f\}$. It can be seen as below.

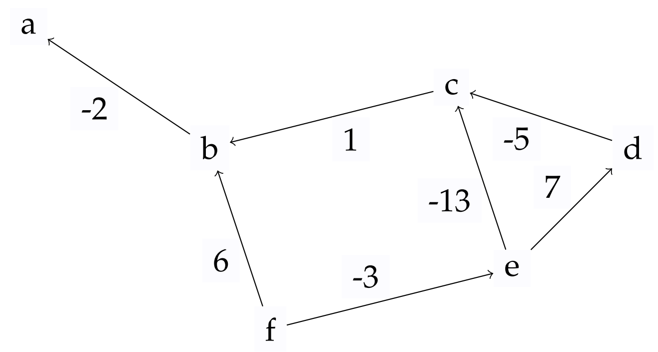


Definition (Directed weight)**.** Given a graph G=(V, E). For any e={a, b} $\in$ E, A directed weight I_e is a function from e to $\mathbb{C}$ such that $I_e(a) + I_e(b) = 0$. If $I_e(a) = I_e(b) = 0$, I_e is called the zero function.

Example 1.14. Let G be the graph of Example 1.2. If we assign a directed weight I_e by $I_{\{a,b\}}(a) = -2, I_{\{a,b\}}(b) = 2, I_{\{b,c\}}(b) = 1, I_{\{b,c\}}(c) = -1, I_{\{c,e\}}(c) = -13, I_{\{c,e\}}(e) = 13$, $I_{\{e,d\}}(e) = 7, I_{\{a,b\}}(d) = -7$, $I_{\{c,d\}}(c) = -5, I_{\{a,b\}}(d) = 5$, $I_{\{e,f\}}(e) = -3, I_{\{e,f\}}(f) = 3$ and $I_{\{b,bf\}}(b) = 6, I_{\{b,f\}}(f) = -6$, then the directed weight graph can be understood as below.

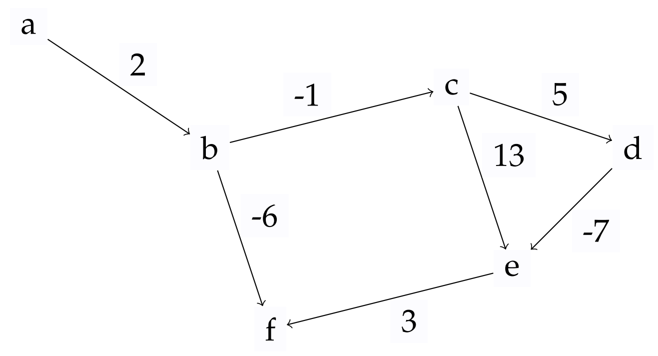


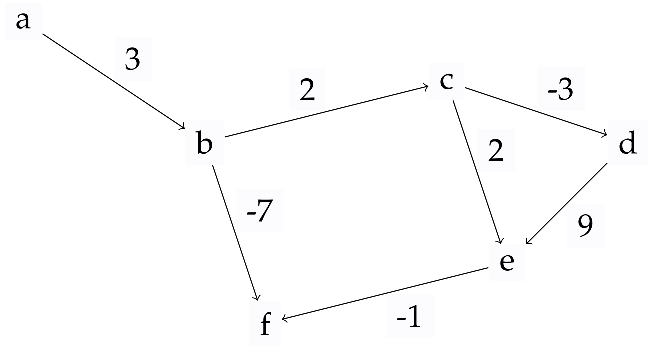
By definition, It can be understood as

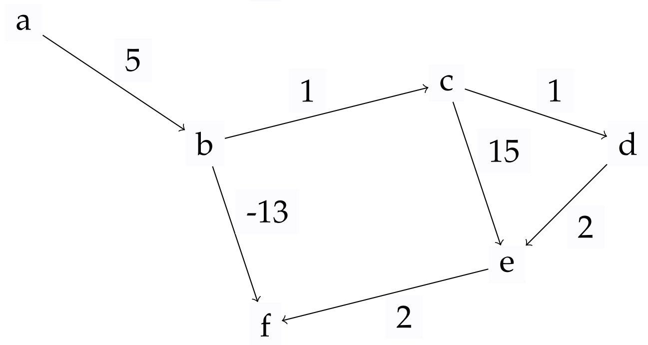


Definition (Operation on directed weight sets)**.** Given a graph G=(V, E). We can consider two sets of directed weight functions $\{I_e\}$ and $\{L_e\}$. Then, define $\{I_e\}$+$\{L_e\}$ as $(I_e$+$L_e)$(v)=I_e(v)+L_e(v) and $\{I_e\}-\{L_e\}$ as (I_e-L_e)(v)=I_e(v)$-L_e$(v). Also, define (aI_e)(v) as $a \cdot (I_e(v))$.

Example 1.15. Let G be the graph of Example 1.2. Suppose following two figures are $\{I_e\}$ and $\{L_e\}$. Then, $\{I_e + L_e\}$ is the last.







1.2 Kirchhoff's Conditions and Spanning Trees

Definition (Kirchhoff's condition)**.** Let G=(V, E) be a weighted graph. Suppose each e ∈ E has a directed weight I_e.

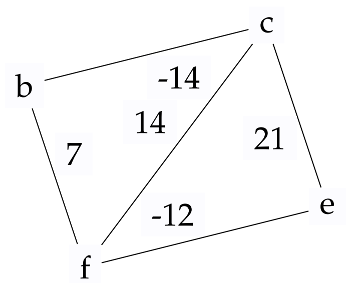
1.(Kirchhoff's current condition) For every v ∈ V, $\sum I_e(v) = 0$.

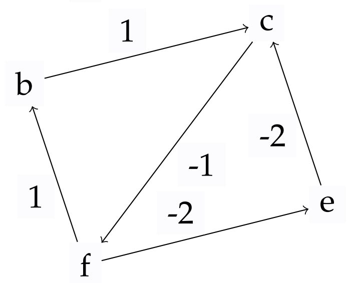
2.(Kirchhoff's voltage condition) For every cycle $\{e_i\}$(i ∈ [n]) in ,

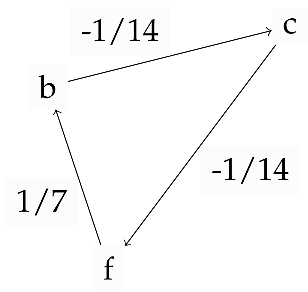
$\sum \frac{(I_{e_i}(a_i)-I_{e_i}(a_{i+1}))}{w_{e_i}} = 0$ (e_i=$\{a_i,\ a_{i+1}\}$, i ∈ [n-1], e_n=$\{a_n,\ a_1\}$, $a_{n+1} = a_1$)

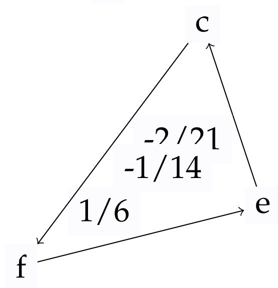
Above two conditions are called Kirchhoff's conditions. If $\{I_e\}$ satisfy Kirchhoff's conditions, we say that $\{I_e\}$ is a current for G.

Example 1.16. Let G and $\{I_e\}$ be a weighted graph and their directed weight as followed. Then, we can verify $\{I_e\}$ satisfy Kirchhoff's conditions. $\sum I_e(b) = 1 + -1 = 0$, $\sum I_e(c) = 1 + 1 + -2 = 0$, $\sum I_e(e) = 2 + -2 = 0$ and $\sum I_e(f) = 1 + 1 + -2 = 0$. And, For each cycles, $\frac{1}{7} + \frac{-1}{14} + \frac{-1}{14} = 0$ and $\frac{-1}{14} + \frac{-2}{21} + \frac{1}{6} = 0$.



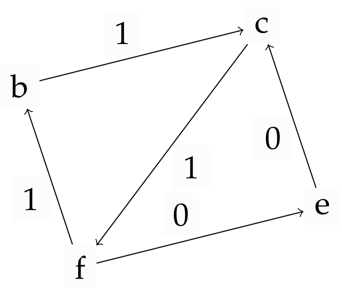


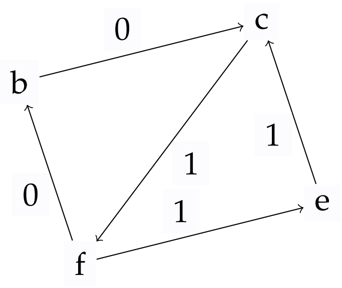




Definition (Directed cycle)**.** Given a graph G=(V, E). For a cycle $\{e_i\}$ in G, if $\{I_{e_i}\}$ satisfy $a = I_{e_i}(a_i) = -I_{e_{i+1}}(a_i) \neq 0$, we call $\{I_{e_i}\}$ as a directed cycle of G with weight a.

Example 1.17. Let G be the weighted graph of Example 1.27. Followings are examples of directed cycles.





Proposition 1.18. *If $\{I_e\}$ satisfy Kirchhoff's current condition, then $\{I_e\}$ is a linear combination of directed cycles.*

Proof. Given a weighted graph G=(V, E) and spanning tree T=(V, L) of G. For each edge e={a, b} ∈ E \ L, the subgraph S=(V, L∪{e}) has a unique cycle C_e since T has a unique path between a and b. Note that the sum of directed cycles C_e satisfy Kirchhoff's current condition. Let's consider $\{I_e\}$ satisfying Kirchhoff's current condition. For each e∈E\L, There exists a directed weight function f_e and a unique cycle C_e. If we regard C_e as a weighted cycle with weight f_e(v), then $\{I_e\}-\sum C_e$ satisfy Kirchhoff's current condition and has zero function at each e∈E and ∉L. It means that $\{I_e\}-\sum C_e$ has possibility of having non-zero directed weight function on e∈T. By Proposition 1.10, there exists a 1-degree vertex v of T. For satisfying Kirchhoff's current condition at v, the directed weight function of v must be zero function. It follows that every weight function of $\{I_e\}-\sum C_e$ are zero function. So, $\{I_e\}=\sum C_e$ □

Example 1.19. Suppose $\{I_e\}$ and $\{L_e\}$ are Example 1.29. Then, we can know that $I_e + L_e$ satisfy Kirchhoff's conditions from Example 1.27.

Remark. Note that the sum of two cycles is also cycle. So, Every cycle in graph G can be represented by a combination of $\{C_e\}$, e$\in E \setminus L$.

Example 1.20. Let G be the graph of Example 1.27 and L=$\{\{b,c\},\{c,f\},\{f,e\}\}$ be a their spanning tree. Then, two cycles C_1 and C_2 of Example 1.29 represent all cycle in G. For example, C=$\{\{b,c\},\{c,e\},\{e,f\},\{b,f\}\}$ is the sum of C_1 and C_2.

Definition (Weighted cycle intersection matrix)**.** Let G=(V, E) be a weighted graph and T=(V, L) be their spanning tree. For e $\in$ E $\setminus$ L, Let's consider a unique cycle C_e. Regard C_e as directed cycle with weight 1 at v. Then, define weighted cycle intersection matrix $D = (d_{i,j}) \in M_{n,n}(n = |\{C_e\}|)$ by $d_{i,j} = \sum_{e \in C_i \cap C_j} \frac{sgn(C_i,C_j)}{w_e}$. Here, if two cycles has the same orientation, $sgn(C_i, C_j) = 1$. Otherwise, $sgn(C_i, C_j) = -1$.

Example 1.21. Let G be the weighted graph of Example 1.27 and their spanning tree T be L of Example 1.33. Also, C_1 and C_2 are two cycles of Example 1.33. Then, $d_{1,1} = \frac{1}{7} + \frac{-1}{14} + \frac{1}{14} = \frac{1}{7}$, $d_{1,2} = d_{2,1} = \frac{1}{14}$ and $d_{2,2} = \frac{1}{14} + \frac{1}{21} + \frac{-1}{12} = \frac{1}{28}$. So, $D = \begin{pmatrix} \frac{1}{7} & \frac{1}{14} \\ \frac{1}{14} & \frac{1}{28} \end{pmatrix}$

Remark. By Proposition 1.17, a current $\{I_e\}$ of graph G must be the linear combinations $\{a_e C_e\}$ of directed cycles. So, We have only to check Kirchhoff's voltage condition. The Kirchhoff's voltage condition about $\{a_e C_e\}$ is $\sum_j a_i d_{i,j} = 0$ for $i \in [n]$. It can be represented by $Da = 0(a = (a_i))$. It means that, there exists a current for G iff $det(D) = 0$.

Example 1.22. Let G be the graph of Example 1.27. Then, from the previous example, We know that their weighted cycle intersection D is $\begin{pmatrix} \frac{1}{7} & \frac{1}{14} \\ \frac{1}{14} & \frac{1}{28} \end{pmatrix}$. Note that $\begin{pmatrix} \frac{1}{7} & \frac{1}{14} \\ \frac{1}{14} & \frac{1}{28} \end{pmatrix}\begin{pmatrix} 1 \\ -2 \end{pmatrix} = 0$.

Definition (weighted spanning trees number)**.** Let G be a weighted graph and w_e be the weight of each edge e. And, $\{T_i\}$ is the set of all spanning tree T_i of G. Then, The weighted spanning trees number of G is defined as $\sum_i \prod_{e \in T_i} w_e$.

Theorem 1.23 (2018, JS)**.** *$\prod w_e \cdot det(D)$ is the weighted spanning trees number.*

Proof.
$\det(D) = \sum_{\sigma \in S_n} sgn(\sigma) \prod_{i=1}^{n} d_{i\sigma(i)} = \sum_{\sigma \in S_n} sgn(\sigma) \prod_{i=1}^{n} \sum_{e_k \in E_{i\sigma(i)}} sgn(e_k) \frac{1}{w_k}$ where E_{ij} is the set of edges in C_i and C_j and w_k is the weight of e_k. We can consider each term a=$sgn(\sigma) \prod_{i=1}^{n} \frac{sgn(C_i, C_{\sigma(i)})}{w_k}$ as subgraph which deletes an edge of each e_k from each $E_{i\sigma(i)}$. Note that It is a necessary condition to make spanning trees. We call such a subgraph of G as a-subgraph. We will show that, for all σ except identity I, a-subgraph in $sgn(\sigma) \prod_{i=1}^{n} d_{i\sigma(i)}$ has a cycle and a is canceled out in $\det(D)$. Consider cycle representation of σ except I. From that, we know that σ has at least one cycle $(i_1 ... i_l)$ which length is larger than one. It means that we delete only common edges in each $f_{i_k} (k \in [l])$ for making a-subgraph from G. So, a-subgraph has a cycle. Since $\prod_{i=1}^{n} d_{ii}$ has every edge, we can get monomial a in $\prod_{i=1}^{n} d_{ii}$. Now, let's compare sgn of two

cases' monomial a. Suppose $f_1, \dots, f_l$ composing a cycle in σ-subgraphs. If all f_i and f_j has opposite direction at E_{ij}, then $sgn(a)$ for a in $\prod_{i=1}^{n} d_{i\sigma(i)}$ is $(-1)^{2m-1}$ $sgn(a)$ for a in $\prod_{i=1}^{n} d_{ii}$. Whenever we change orientation of a cycle, their sgn change by $(-1)^2$. So, It's not depending on cycles' orientation. Now, we know that every monomials in σ-subgraphs are canceled with the corresponding case in $\prod_{i=1}^{n} d_{ii}$. Also, we can know that remaining monomial a in $\prod_{i=1}^{n} d_{ii}$ have bijection with every spanning tree(If some a appears more than once in $\prod_{i=1}^{n} d_{ii}$, It means that subgraph of the monomial a has cycle so that a appears in some $\sigma \neq I$). So, $\det(D) = \sum_S \prod_{e_i \notin S} \frac{1}{w_i} = \frac{\sum_S \prod_{e_i \in S} w_i}{\prod_{e_i \in E} w_i}$. □

Corollary 1.24 (2018, JS). *There exists a current for G iff the weighted spanning trees number of G is zero.*

Proof. Note that, the number of weighted spanning trees of G is zero if and only if $\det(D) = 0$. Also, $\det(D) = 0$ if and only if there exists x such that $Dx = 0$ from linear algebra. Since such x is a current, the corollary is proved. □

Chapter 2

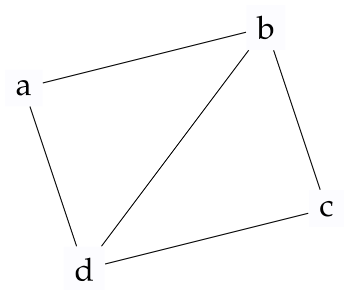
Effective Resistance and Kirchhoff's Conditions

In this chapter, we will introduce resistor network theory. Resistor network theory deal with non-negative weighted graph. By above section, we can generalize resistor network theory and define effecitve resistance.

2.1 Effective Resistance

Definition (Adjacency matrix)**.** Given a weighted graph G=(V, E). Let $|V|$ be n. The adjacency matirx $A(G) \in M_n$ of G is defined as $a_{ij} = w_{ij}$ for $i \neq j$ and $a_{ii} = 0$.

Example 2.1. Suppopse a graph G=(V, E) where V=$\{a, b, c, d\}$ and E=$\{\{a, b\}, \{b, c\}, \{c, d\}, \{d, a\}, \{b, d\}\}$. If we give weight 1 to each edge, then $A(G) = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$



Definition (Diagonal matrix)**.** Given a graph G=(V, E). Let $|V|$ be n. The diagonal matrix $D(G) \in M_n$ of G is defined as $a_{ij} = 0$ for $i \neq j$ and $a_{ii} = d_i$ where d_i is the degree of vertex i.

Example 2.2. Let G be the graph of Example 2.2. Then, their diagonal matrix $Dia(G) = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$

Definition (Laplacian matrix)**.** Let G=(V, E) be a weighted graph. Then, the Laplacian matrix $L(G) \in M_n$ of G is defined as $L(G) = Dia(G) - A(G)$.

Example 2.3. Let G be the weigthed graph of Example 2.2. Then their laplacian matrix $L(G) = Dia(G) - A(G) = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} - \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} =$

$\begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 3 & -1 & -1 \\ 0 & -1 & 2 & -1 \\ -1 & -1 & -1 & 3 \end{pmatrix}$

Definition (Combinatorial green's function)**.** Let G=(V, E) be a weighted graph. the Combinatorial green's function $\mathcal{G}$ of G is $(L(G) + J)^{-1}$ where J is each of whose entries 1 $n \times n$ matrix.

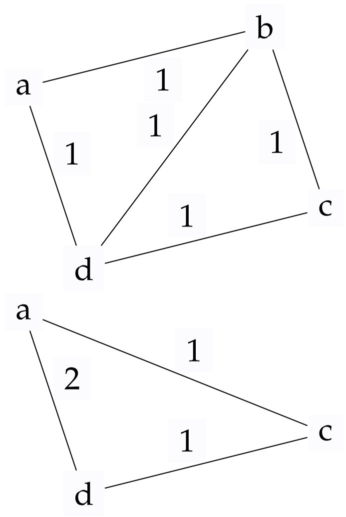
Example 2.4. Let G be the weighted graph of Example 2.2. Then, the combinatorial green's function $\mathcal{G}$ is

$$(L(G) + J)^{-1} = \begin{pmatrix} 3 & 0 & 1 & 0 \\ 0 & 4 & 0 & 0 \\ 1 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}^{-1} = \frac{1}{8} \begin{pmatrix} 3 & 0 & -1 & 0 \\ 0 & 2 & 0 & 0 \\ -1 & 0 & 3 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

Remark. Note that we cannot define combinatorial green's function when $L(G) + J$ has no inverse. So, in resistor network theory, we consider only non-negative weighted graph.

Now, we will review several linear algebra for introducing classical definition of effective resistance.

Definition (Contraction of a weighted graph)**.** Given a weighted graph G=(V, E). Then, An minor of a graph G with respect to a and b is called contraction G_{a*b} of G that is obtained by shrinking(contracting) the edge ab to a point. We may contract ab by identifying the vertex a with the vertex b so that all edges that were incident to a and b in G are incident to b in G_{a*b}(and a is no longer a vertex of G_{a*b}). As a weighted graph, we define the contraction G_{a*b} of G to be a weighted graph satisfying the following conditions. Let w and ω denote the weights, and d and δ denote the degrees in G and G_{a*b}, respectively. Then, $V(G_{a*b}) = V(G) \setminus \{a\}$, $\omega_{ib} = w_{ia} + w_{ib}$ and $\omega_{ij} = w_{ij}$ if $i, j \neq b$. $\delta_b = d_a + d_b$, and $\delta_i = d_i$ if $i \neq b$.



Although the roles of a and b may be switched in defining G_{a*b}, we will not do so in order to avoid confusion in what follows. It is important to note that G_{a*b} is defined as a wieghted graph even when $w_{ab} = 0$ in G. Hence, we may have $k(G_{a*b}) > k(G)$. For example, if G has three vertex a, b, c and $w_{ac} = w_{bc} = 1$ and $w_{ab} = 0$, then $k(G_{a*b}) = 2$ and $k(G) = 1$. Since a weighted graph can be recovered from its Laplacian matrix, we may also define G_{a*b} to be the graph whose Laplacian matrix is obtained from $L = L(G)$ by applying the following sequence of operations:($R_i(M)$ and $C_j(M)$ denote row i and column j of a matrix M, respectively)

1. replace $R_b(L)$ by $R_a(L) + R_b(L)$(denote the result by M_1).

2. replace $C_b(M_1)$ by $C_a(M_1) + C_b(M_1)$(denote the result by M_2)

3. delete $R_a(M_2)$ and $C_a(M_2)$(denote the result by M_3).

Clearly, the entries in M_2 except those in $R_a(M_2)$ and $C_a(M_2)$ are the weights and degrees for G_{a*b} as described above. The last operations corresponds to eliminating the vertex a from $V(G)$. It is clear that we have $M_3 = L(G_{a*b})$.

Note that the above three operations may be applied to any square matrix M, and we will denote the resulting matrix by M_{a*b}, called a contraction of M. With this notation, we have $L(G_{a*b}) = L(G)_{a*b}$. we note the following useful properties of contractions. If M has the zero sum property for its rows and columns, then so does any contracation of M. If M has rank ≤ 1, then so does any contraction of M. The sum of all entries of M equals that of any contraction of M. We also have

$$(M+N)_{a*b} = M_{a*b} + N_{a*b}.$$

Example 2.5. This example will be discussed again. Let G be a weighted graph with the vertex set $\{1,2,3,4\}$ with the weights $w_{13} = 0$, $w_{24} = 2$, and $w_{ij} = 1$ otherwise. Then, $L(G) = \begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 4 & -1 & -2 \\ 0 & -1 & 2 & -1 \\ -1 & -2 & -1 & 4 \end{pmatrix}$. The followings are the Laplacian matrices for various contractions of G: $L(G_{1*3}) = \begin{pmatrix} 4 & -2 & -2 \\ -2 & 4 & -2 \\ -2 & -2 & 4 \end{pmatrix}$, $L(G_{2*4}) = \begin{pmatrix} 2 & 0 & -2 \\ 0 & 2 & -2 \\ -2 & -2 & 4 \end{pmatrix}$, $L(G_{1*2}) = \begin{pmatrix} 4 & -1 & -3 \\ -1 & 2 & -1 \\ -3 & -1 & 4 \end{pmatrix}$.

Lemma 2.6. *Let M=(m_{ij}) be an n by n matrix, and let μ_{ij} be the (i, j)-cofactor of M, i.e., $\mu_{ij} = (-1)^{i+j} \det M_{ij}$, where M_{ij} is obtained from M by removing $R_i(M)$ and $C_j(M)$. The following identity holds for all i and j:* $\det M_{i*j} = \mu_{ii} + \mu_{jj} - \mu_{ij} - \mu_{ji}$.

Proof. First, we will prove the case $i = 1$ and $j = 2$. Note that M_{11} and M_{12} differ only in their first columns. Also, M_{22} and M_{21} differ only in their frist columns. By the linearity of determinant on columns, we have $\mu_{11} - \mu_{12} = \det M_{11} + \det M_{12} = \det M'$ and $\mu_{22} - \mu_{21} = \det M_{22} + \det M_{21} = \det M''$, where $M' = \begin{pmatrix} m_{22}+m_{21} & m_{23} & \dots & m_{2n} \\ m_{32}+m_{31} & m_{33} & \dots & m_{3n} \\ \dots & \dots & \dots & \dots \\ m_{n2}+m_{n1} & m_{n3} & \dots & m_{nn} \end{pmatrix}$ and

$$M'' = \begin{pmatrix} m_{11}+m_{12} & m_{13} & \dots & m_{1n} \\ m_{31}+m_{32} & m_{33} & \dots & m_{3n} \\ \dots & \dots & \dots & \dots \\ m_{n1}+m_{n2} & m_{n3} & \dots & m_{nn} \end{pmatrix}.$$

Note that M' and M'' differ only in their first rows. It is now clear the linearity of determinant on rows that $\det M' + \det M'' = \det M_{1*2}$.

In general, assume $1 \leq i < j \leq n$. Let P be the matrix obtained from M by switching row i with row 1, row j with row 2, column i with column 1, and column j with column 2. Let π_{ij} be the (i, j)-cofactor of P. Then, $\pi_{11} = \mu_{ii}$ because P_{11} and M_{ii} differ by an even permutation of rows and columns. Similarly, we have $\pi_{22} = \mu_{jj}$, $\pi_{12} = \mu_{ij}$ and $\pi_{21} = \mu_{ji}$. Also it is easiy checked that M_{i*j} and P_{1*2} differ by an even permutation of rows and columns. Therefore, $\det M_{i*j} = \det P_{1*2}$, and the result follows. ☐

If an n by n matrix M satisfies the zero sum condition for its rows and columns, then it is easily checked that every cofactor of M has the same value. For example, we've already seen that every cofactor of the Laplacian matrix $L(G)$ of finite graph G is $k(G)$. The following is a generalization of Temperley's tree-number formula.

Theorem 2.7. *Let M be an n by n matrix such that the sum of entries in every row and every column is zero, and let μ denote the value of any cofactor of M. Let U be an n by n rank 1 matrix, and let σ denote the sum of all of its entries. Then the following identity holds: $\mu * \sigma = \det(M + U)$.*

Proof. Let $M + U = (C_1 + D_1, C_2 + D_2, ..., C_n + D_n)$ where C_i and D_i are the i column of M and U, respectively. Given any subset $S \subset [n]$, define $\triangle_S = (X_1, X_2, ..., X_n)$, where $X_i = D_i$ if $i \in S$ and $X_i = C_i$ if i$\notin S$. By the multilinearity of determinant on columns, we have $\det(M + U) = \sum_{S\in[n]} \det \triangle_S$ where the sum is over all subsets S of $[n]$. Clearly, we have $\det \triangle_\emptyset = \det M = 0$. Also, if $|S| > 1$, then $\det \triangle_S = 0$ because U has rank 1 and every column of U is a multiple of a single column. Futhermore, if we let σ_i be the sum of all entries in D_i, then $\det_{\triangle_{\{i\}}} = \mu * \sigma_i$ for every $i \in [n]$. Therefore, we have $\det(M + U) = \sum_{0\leq i\leq n} \det_{\triangle_{\{i\}}} = \sum_{0\leq i\leq n} \mu * \sigma_i = \mu * \sigma$. □

We already know that the value of any cofactor of $L(G)$ is $k(G)$.

Corollary 2.8. *For a finite weighted graph G with n vertices, we have $n^2 * k(G) = \det L(G)$.*

Theorem 2.9. *Let G be a finite weighted graph with n vertices and $\kappa(G) \neq 0$. The entries in $\mathcal{G} = (g_{ij})$ satisfy the following identities:*
$g_{aa} + g_{bb} - g_{ab} - g_{ba} = \frac{\kappa(G_{a*b})}{\kappa(G)}$ for any arbitrary pair of distinct vertices a and b of G.

Proof. Let l_{ij} be the (i, j)-cofactor of L. Note that we have $g_{ij} = l_{ij}/\deg L$ because $G = L^{-1}$. We also have $\det L = n^2 k(G)$ by Corollay 3. Therefore, $g_{aa} + g_{bb} - g_{ab} - g_{ba} = \frac{1}{n^2 k(G)}(l_{aa} + l_{bb} - l_{ab} - l_{ba}) = \frac{1}{n^2 k(G)} \det L_{a*b} = \frac{1}{n^2 k(G)} \det(L(G)_{a*b} + J_{a*b})$. Since J has rank 1 and the sum of its entries is n^2, the same is true for J_{a*b}. Also we have $L(G)_{a*b} = L(G_{a*b})$, and every cofactor of $L(G_{a*b})$ equals $k(G_{a*b})$. Therefore, by above theorem, we have $\det(L(G)_{a*b} + J_{a*b}) = n^2 k(G_a)$, and the theorem follows. □

Definition (Effective resistance)**.** Let G=(V, E) be a weighted graph. For each a ∈ V, let V_a be the electrical potential at vertex a and I_a be the net current flowing into the graph with $\sum I_a = 0$. The Kirchhoff's law states $\sum_j w_{ij}(V_i - V_j) = I_i$ for each i, which is equivalent to LV=I where $V = (V_i)$ and $I = (I_i)$. Suppose external current source with current I is connected to a and b so that $I_a = I$, $I_b = -I$ and $I_i = 0$ for $i \neq a, b$.Then, Effecitve resistance $R_{a,b}$ between a and b is defined by $R_{a,b} = \frac{V_a - V_b}{I}$.

In physics, we can compute effective resistance by kirchhoff's laws. Also, we can compute effective resistance by combinatorial green's function. For that, we will introduce a lemma.

Lemma 2.10. *Suppose $y \perp 1$. Then, $x = Gy$ is a solution to $Lx = y$.*

Proof. Let $x = Gy$. Then we have $Lx = y$. Multiplying this equation by J, we get $JLx + J^2 x = Jy$. Since $JLx = Jy = 0$ and $J^2 = nJ$, we see that $Jx = 0$. Therefore $Lx = (L + J)x = Lx = y$. □

Since we have $\overrightarrow{I} \perp 1$, by above lemma, $\overrightarrow{V} = G\overrightarrow{I}$ is a solution to $L\overrightarrow{V} = \overrightarrow{I}$. Therefore, we have $V_a = (g_{aa} - g_{ab})I$ and $V_b = (g_{ba} - g_{bb})I$. So, below property follows.

Proposition 2.11. *Given a positive weighted graph G=(V, E).* $R_{a,b}(G) = \frac{\kappa(G_{a*b})}{\kappa(G)}$.

Above deal with the effective resistance of non-negative weighted graph G. Now, we are ready to define generalized effective resistance for a weighted graph. Given a weighted graph G=(V, E). Note that

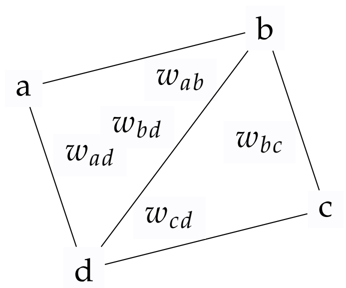
$$-C_{a,b}(G)\kappa(G_{a*b}) + \kappa(G) = (-C_{a,b} + w_{a,b})\kappa(G_{a*b}) + \kappa(G - \{a, b\}) = 0$$

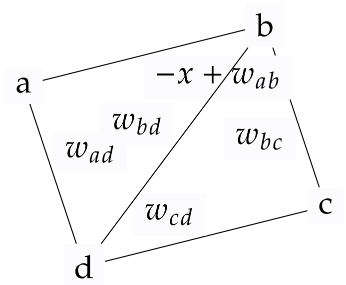
by above propositions. Also, we know that there exists a current iff the number of spanning trees of G is zero. So, The meaning of effective conductance $C_{a,b}$ is the weight of external current source connected to a and b which makes the number of spanning trees zero.

2.2 Generalized Effective Resistance in Graphs

Theorem 2.12. *The effective conductance $C_{a,b}$ between a and b is the solution of $det(D'(x)) = 0$ where $D'(x)$ is defined by replacing $w_{a,b}$ in D by $-x + w_{a,b}$.*

Example 2.13. Let G be the weighted graph of Example. Then, $C_{a,b}$ is the solution of $\det(D'(x)) = 0$ where $D'(x) = \begin{pmatrix} \frac{1}{-x+w_{ab}} + \frac{1}{w_{bd}} + \frac{1}{w_{ad}} & \frac{1}{w_{bd}} \\ \frac{1}{w_{bd}} & \frac{1}{w_{bc}} + \frac{1}{w_{cd}} + \frac{1}{w_{bd}} \end{pmatrix}$.





Definition. Given weighted graph G=(V, E). For any $a, b \in V$, The effecitve conductance $C_{a,b}$ of G is defined by the solution of $det(D'(x))$.

This definition is meaningful for general weighted graph since we can compute determinant of weighted cycle intersection matrix of every weighted graph G.

Chapter 3

On Higher Dimensional Spaces

In this chapter, we will review simplicial complex and homology theory. And then, we will introduce another definition of weighted cycle intersection matrix so that we can generalize our graph theory to simplicial complex.

3.1 Preliminaries

Definition (Simplex)**.** In $\mathbb{R}^n$, A k-simplex is a k-dimensional polytope which is the convex hull of its k+1 vertices.

More formally, suppose the k+1 points $u_0, ..., u_k \in \mathbb{R}^n$ are affinely independent, which means $u_1 - u_0, ..., u_k - u_0$ are linearly independent. Then, the simplex determined by them is the set of points $C = \{ \theta_0 u_0 + \cdots +$

$\theta_k u_k \mid \sum_{i=0}^{k} \theta_i = 1,\ \theta_i \geq 0\}$. We will denote C as $(u_0, \dots, u_k)$. In this case, The orientation of simplex is given by orderings of the vertices. Two orderings define the same orientation if and only if they differ by an even permutation. So, every simplex has exactly two orientations.

Definition (Face of simplex)**.** Given a n-simplex C. Then, The convex n hull of a subsets of n+1 vertices is m-simplex, called an m-face of the n-simplex.

Definition (Simplicial complex)**.** A simplicial complex $\mathbb{K}$ is a set of simplices that satisfies the following conditions:

1. Every face of a a simplex from $\mathbb{K}$ is also in $\mathbb{K}$.

2. The intersection of any two simplices $\sigma_1,\ \sigma_2\ \in \mathbb{K}$ is a face of both σ_1 and σ_2.

A simplicial k-complex is a simplicial complex where the largest dimension of any simplex in $\mathbb{K}$ equals k. A homogeneous simplicial k-complex $\mathbb{K}$ is a simplicial complex where every simplex of dimension less then k is a face of some simplex $\sigma\ \in \mathbb{K}$ of dimension k.

A facet is any simplex in a complex that is not a face of any larger dimensional simplex.

Let S be a simplicial complex. A simplicial k-chain is a finite formal sum $\sum_{i=1}^{N} c_i \sigma_i$, where c_i is an integer and σ_i is an oriented k-simplex. In this definition, we declare that each oriented simplex is equal to the negative of the simplex with the opposite orientation. For example, $(v_0,\ v_1) = -(v_1,\ v_0)$.

The group of k-chains on S is written C_k. This is a free abelian group which has a basis in one to one correspondence with the set of oriented k-simplices in S.

Definition (Boundary operator)**.** Given simplicial complex S. Let $\sigma = (u_0, \ldots, u_k)$ be an oriented k-simplex, a basis element of C_k in S. The boundary operator $\partial_k : C_k \to C_{k-1}$ is homomorphism defined by:

$\partial_k(\sigma) = \sum_{i=0}^{k}(-1)^i\,(u_0, \ldots, \hat{u}_i, \ldots, u_k)$, where $(u_0, \ldots, \hat{u}_i, \ldots, u_k)$ is i-th face of σ, obtained by deleting its i-th vetex.

In C_k, elements of the subgroup $Z_k = ker\partial_k$ are referred to as cycles, and the subgroup $B_k = Im\partial_{k+1}$ is said to consist of boundaries. It is easy to check $\partial^2 = 0$. So, the abelian group (C_k, ∂_k) form a chain complex. The k-homology group H_k of S is defined to be the quotient abelian group $H_k(S) = \frac{Z_k}{B_k}$.

Definition (Dimension of simplicial complex)**.** Let $\mathbb{K}$ be a simplicial complex. For $\sigma \in \mathbb{K}$, the dimension of σ is defined as $\dim(\sigma) = |\sigma| - 1$. Then, $\dim(\mathbb{K}) = \max\{\dim(\sigma) \mid \sigma \in \mathbb{K}\}$.

Definition (i-skeleton)**.** Let $\mathbb{K}$ be a simplcial complex and X_i denote the collection of i-dimensional simplicies(i-faces). The i-skeleton of X is $X^{(i)} = \cup_{0 \le j \le i} X_j$.

Definition (Simplicial network)**.** we define a $simplicial\,network$ to be a simplicial complex $\mathbb{K}$ of positive dimension d where each d-dimensional simplex τ is assigned a $resistance$ r_τ and weighted by its $conductance$ $c_\tau := \frac{1}{r_\tau}$.

Definition (Resistance matrix). The $resistance\ matrix\ R$ is a square diagonal matrix whose diagonal entries are r_τ.

3.2 Kirchhoff's Conditions and Spanning Trees in Higher Dimensional Spaces.

Definition (i-dimensional spanning tree). Given a d-dimensional simplicial complex S. For a non-empty subset $T \subset S$, define $S_T = T \cup X^{i-1}$ to be an i-dimensional subcomplex of S. For $i \in [-1, d]$, a subcomplex X_T of X is an i-dimensional spanning tree if

1. $\tilde{H}_i(S_T) = 0$.

2. $\mathrm{rk}\tilde{H}_{i-1}(S_T) = 0$.

Denote the set of T for all i-dimensional spanning trees S_T of S by $T_i := T_i(X)$. Note that $|\tilde{H}_i(X_T)|$ is finite for an i-dimensional spanning tree S_T, define the i-th spanning tree number $k_i(S)$ of S to be $k_i(S) := \sum_{T \in T_i} |\tilde{H}_{i-1}(S_T)|^2$. This definition generalizes the spanning trees number of a graph.

Proposition 3.1. *Given a simplicial complex S. For $T \in S_d$, let $\partial_{\bar{T}}$ be the submatrix of ∂ obtained by deleting the rows indexed by T. Then, $\partial_{\bar{T}}$ is a non-singular square matrix iff $T \in T_d(S)$, and in that case, $|det\partial_{\bar{T}}|$=$|\tilde{H}_{d-1}(S_T)|$.*

So, we can obtain a formula for $k_d(S)$ with ∂, $k_d(S) = \sum_{T \in T_d(S)} (det \partial_{\bar{T}})^2 = det \partial^t \partial$ where the second equality follows from the Cauchy-Binet Theorem.

Recall that each top-dimensional simplex τ of a network (S, R) is weighted by its conductance $c_\tau = \frac{1}{r_\tau}$. For non-empty $T \subset S_d$, let $c_T = \prod_{\tau \in T} c_\tau$. We define the weighted tree-number $\hat{k_d}(S)$ of (S, R) to be

$\hat{k_d}(S) := \sum_{T \in T_d(S)} c_T |\tilde{H}_{i-1}(X_T)|^2 = \det R^{-1} \det \partial^t_{d+1} R \partial_{d+1}|_{basis\, of\, \ker \partial_d}$.

Note that, On graphs, Currents are x such that $Dx = 0$ where weighted cycle intersection matrix D. In fact, we can verify $D = \det \partial^t_2 R \partial_2|_{basis\, of\, \ker \partial_1}$. So, we can generalize $i-dimensional\ current$ by kernel of $\det \partial^t_{i+1} R \partial_{i+1}|_{basis\, of\, \ker \partial_i}$. By above, the weighted i-dimensional tree number is
$\hat{k_i}(S) = \det R^{-1} \det \partial^t_{i+1} R \partial_{i+1}|_{basis\, of\, \ker \partial_i}$. Consequently, there exists a current which satisfy d-dimensional Kirchhoffs' laws if and only if $\hat{k}_d(S) = 0$ as same as graphs' case. So, we can define the effective resistance R_σ of σ as the solution x of $\det \partial^t_{i+1} R \partial_{i+1}|_{basis\, of\, \ker \partial_i} = 0$.

Bibliography

[1] N. Biggs, Algebraic Graph Theory, second ed., Cambridge University Press, Cambridge, 1993.

[2] D.Kim, *Critical phenomena on fractal lattices*, in: Chin-Kun Hu (Ed.), Progress in Statistical Mechanics, World Scientific Pub. Co., Singapore, 1988, pp. 71-80.

[3] R.Stanley, Enumertaive Combinatorics II, Cambridge University Press, Cambridge, 1999.

[4] G. Strang, Introduction to Applied Mathematics, Cambridge University Press, Wellesley, 1986.

[5] K. Stephenson, M. Zelen, Rethinking centrality: Methods and examples, Social Networks 11 (1989) 1-37.

[6] H.N.V. Temperley, On the mutual cancellation of cluster integrals in Mayer's fugacity series, Proc. Phys. Soc. 83 (1964) 3-16.

[7] C. Thomassen, Resistances and currents in infinite electrical networks, J. Combin. Theory Ser. B 49 (1990) 87-102.

[8] D. West, Introduction to Graph Theory, Prentice Hall, 1996.

[9] F.Y. Wu, Theory of resistor network: The two point resistance, J. Phys. A 37 (2004) 6653-6673.

[10] G. Aalipour, A.M. Duval, W. Kook, K. Lee, and J. L. Martin, A weighted cellular matrix-tree theorem, with applications to complete colorful and cubical complexes, preprint.

[11] R. Adin, Counting colorful multi-dimensional trees, Combinatorica 12(3) (1992), 247-260.

[12] N. Biggs, Algebraic potential theory on graphs, Bull. London Math. Soc. 29(6) (1997) 641682.

[13] M. Catanzaro, V. Chernyak and J. Klein, Kirchhoff's theorems in higher dimensions and Reidemeister torsion, Homology, Homotopy and Applications 17(1) (2015) 165-189.

[14] A. Duval, C. Klivans and J. Martin, Simplicial matrix-tree theorems, Trans. Amer. Math. Soc. 361(2009) 6073-6114.

[15] A. Duval, C. Klivans and J. Martin, Cellular spanning trees and Laplacians of cubical complexes, Adv. Appl. Math. 46 (2011) 247-274.

[16] B. Eckmann, Harmonishce Funktionen und Randwertaufgaben in einem Komplex, Commentarii mathematici Helvetici 17 (1944) 240-255.

[17] R. Foster, The average impendance of an electrical network, In Contributions to Applied Mechanics (Reissner Anniversary Volume) (1949) 333-340.

[18] J. Friedman, Computing Betti numbers via combinatorial Laplacians, Algorithmica 21 (1998) 331-346.

[19] F. R. Gantmacher, The Theory of Matrices- Vol I, Chelsea, New York, 1960.

[20] A. Hatcher, Algebraic Topology, Cambridge University Press, 2001.

[21] X. Jiang, L. Lim, Y. Yao and Y. Ye, Statistical ranking and combinatorial Hodge theory. Mathematical Programming, 127(1) (2011) 203-244.

[22] G. Kalai, Enumeration of Q-acyclic simplicial complexes, Israel J. Math. 45 (1983) 337-351.

[23] H. Kim and W. Kook, Logarithmic tree-numbers for acyclic complexes, Electron. J. Combin. 21 (2014) P1.50, 12 pp.

[24] W. Kook, Combinatorial Green's function of a graph and applications to networks, Adv. Appl. Math. 46 (2011) 417-423.

[25] L. Lim, Hodge Laplacians on graphs. arXiv preprint arXiv:1507.05379 (2015).

[26] J. Matouoek, Using the Borsuk-Ulam Theorem: Lectures on Topological Methods in Combinatorics and Geometry, Springer, 2003.

[27] J. Munkres, Elements of Algebraic Topology, Addison-Wesley, Reading, MA, 1984.